Nonlinear Dynamics, Psychology, and Life Sciences, Vol. 9, No. 4, October, 2005
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Methodological issues in the application of monofractal analyses in psychological and behavioral research

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Abstract: *A number of recent research works tried to apply fractal methods to psychological or behavioral variables. Quite often, nevertheless, the use of fractal analyses remains rudimentary, and the goal of researchers seems limited to evidencing the presence of long-range correlation in data sets. This article presents some recent developments in monofractals theory, and some related methodological refinements. We also discuss a number of specific issues related to the application of fractal methods in psychological and behavioral research. Finally, we consider the potential use of such approach for a renewal of classical issues in psychology and behavioral science.*

Key Words: Fractal analysis, fractional Gaussian noise, fractional Brownian motion, Hurst exponent, series length

INTRODUCTION

A number of experimental papers, in the last decade, revealed the fractal properties of psychological or behavioral variables, when considered from the point of view of their evolution over time. Fractals were evidenced, for example, in self-esteem (Delignières, Fortes, & Ninot, 2004), in mood (Gottschalk, Bauer, & Whybrow, 1995), in serial reaction time (Gilden, 1997; van Orden, Holden, & Turvey, 2003), in

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finger tapping (Gilden, Thornton, & Mallon, 1995; Delignières, Lemoine, & Torre, 2004), in stride duration during walking (Hausdorff, Peng, Ladin, Wei, & Goldberger, 1995), in relative phase in a bimanual coordination task (Schmidt, Beek, Treffner, & Turvey, 1991; Torre, Lemoine & Delignières, 2004), and in the displacement of the center-of-pressure during upright stance (Collins & De Luca, 1993; Delignières, Deschamps, Legros, & Caillou, 2003). Generally, these variables were previously conceived as highly stable over time, and fluctuations in successive measurements were considered as randomly distributed, and uncorrelated in time. As such, a sample of repeated measures was assumed to be normally distributed around its mean value, and noise could be discarded by averaging. This methodological standpoint was implicitly adopted in most classical psychological researches (for a deeper analysis, see Gilden, 2001; Slifkin & Newell, 1998). In other words, temporal ordering of data points was ignored and the possible correlation structure of fluctuations was clearly neglected.

Fractal analysis focuses in contrast on the time-evolutionary properties of data series and on their correlation structure. Fractal processes are characterized by a complex pattern of correlations appearing following multiple interpenetrated time scales. In such process, the value at a particular time is related not just to immediately preceding values, but to fluctuations in the remote past. Fractal series are also characterized by self-similarity, signifying that the statistical properties of segments within the series are similar, whatever the time scale of observation.

Evidencing fractal properties in empirical time series has important theoretical implications, and could lead to a deep renewal of models. Fractals are considered as the natural outcome of complex dynamical systems behaving at the frontier of chaos (Bak & Chen, 1991; Marks-Tarlow, 1999). Psychological variables should then be conceived as the macroscopic and dynamical products of a complex system composed of multiple interconnected elements. Moreover, psychological and behavioral time series often present fractal characteristics close to a very special case of fractal process, called $1/f$ or *pink* noise.'$1/f$ noise' signifies that when the power spectrum of these time series is considered, each frequency has power proportional to its period of oscillation. As such, power is distributed across the entire spectrum and not concentrated at a certain portion. Consequently, fluctuations at one time scale are only loosely correlated with those of another time scale. This relative independence of the underlying processes acting at different time scales suggests that a localized perturbation at one time scale will not necessarily alter the stability of the global system. In other words, 1/*f* noise renders the system more stable and more adaptive to internal and external perturbations (West & Shlesinger, 1989).

1/*f* noise was evidenced in most series produced by "normal" participants, characterized as young and healthy. As such, this 1/*f* behavior could be considered as an indicator of the efficiency of the system which produced the series. In contrast, series obtained with older participants or with patients with specific pathologies exhibited specific alterations in fractality (Gottschalk et al., 1995; Hausdorff et al., 1997; Yoshinaga, Miyazima, & Mitake, 2000).

The application of fractal methods, nevertheless, often remains rudimentary: analyses are limited to the use of a unique method, the collected series are sometimes too short for a valid assessment, and more generally the theoretical background of fractals and related methods is not fully exploited. The recent theoretical and methodological refinements of fractal analyses (see, for example Eke et al., 2000; Eke, Hermann, Kocsis, & Kozak, 2002) appear largely unknown in the psychological community. These methodological limitations severely restrain the potential impact of fractal approaches on psychological theories and models, and frequently the discovery of a fractal behavior in experimental series is just presented as an anecdotic result.

The aim of this paper is to present a wide overview of fractal analyses, with a special focus on the recent refinements of both theoretical background and methodological approach. We first develop the formal distinction between fractional Gaussian noise and fractional Brownian motion, which seems essential for relevant application of fractal methods. Then we present the different methods proposed in the literature, in the frequency and the time domains. The next section will focus on the methodological problems related to the identification of long-range correlations, the classification of series, and the estimation of the fractal exponents. Finally, we evoke some more basic questions, concerning the nature of the empirical series to use in a fractal approach, and we suggest some guidelines for the development of such approach in psychological and behavioral research.

THE FGN/FBM MODEL

In order to ensure better understanding of the following parts of this article, a deeper and more theoretical presentation of fractal processes is necessary. A good starting point for this presentation is Brownian motion, a well-known stochastic process that can be represented as the random movement of a single particle along a straight line. Mathematically, Brownian motion is the integration of a white Gaussian noise. As such, the most important property of Brownian motion is that its successive increments in position are uncorrelated: each displacement is independent of the former, in direction as well as in amplitude. Einstein (1905) showed that, on average, this kind of motion moves a particle from its origin by a distance that is proportional to the square root of the time.

Mandelbrot and van Ness (1968) defined a family of processes they called *fractional Brownian motions* (fBm). The main difference with ordinary Brownian motion is that in an fBm successive increments are correlated. A positive correlation signifies that an increasing trend in the past is likely to be followed by an increasing trend in the future. The series is said to be persistent. Conversely, a negative correlation signifies that an increasing trend in the past is likely to be followed by a decreasing trend. The series is then said to be anti-persistent.

Mathematically, a fBm is characterized by the following scaling law:

$$<\Delta x> \propto \Delta t^H \quad (1)$$

which signifies that the expected displacement $<\Delta x>$ is a power function of the time interval (Δt) over which this displacement is observed. H represents the typical scaling exponent of the series and can be any real number in the range $0 < H < 1$. The aims of fractal analysis are to check whether this scaling law holds for experimental series and to estimate the scaling exponent. Ordinary Brownian motion corresponds to the special case $H = 0.5$ and constitutes the frontier between anti-persistent ($H < 0.5$) and persistent fBms ($H > 0.5$).

Fractional Gaussian noise (fGn) represents another family of fractal processes, defined as the series of successive increments in an fBm. Note that fGn and fBm are interconvertible: when an fGn is cumulatively summed, the resultant series constitutes an fBm. Each fBm is then related to a specific fGn, and both are characterized by the same H exponent. These two processes possess fundamentally different properties: fBm is non-stationary with time-dependent variance, while fGn is a stationary process with a constant expected mean value and constant variance over time. The H exponent can be assessed from an fBm series as well as from the corresponding fGn, but because of the different properties of these processes, the methods of estimation are necessarily different.

Recently a systematic evaluation of fractal analysis methods was undertaken by Bassingthwaighte, Eke, and collaborators (Caccia, Percival, Cannon, Raymond, & Bassingthwaighte, 1997; Cannon, Percival, Caccia, Raymond, & Bassingthwaighte, 1997; Eke et al., 2000, 2002). This methodological effort was based on the previously described dichotomy between fGn and fBm. According to these authors, the first step in a fractal analysis aims at identifying the class to which the analyzed series belongs, i.e. fGn or fBm. Then the scaling exponent can be properly assessed, using a method relevant for the identified class. The evaluation proposed by these authors clearly showed that most methods gave acceptable estimates of H when applied to a given class (fGn or fBm), but led to inconsistent results for the other. As claimed by Eke et al. (2002), researchers were not aware before a recent past of the necessity of this dichotomic model. As such, a number of former empirical analyses and theoretical interpretations remain questionable.

MONOFRACTAL ANALYSIS METHODS

We present in this section the most commonly used methods in monofractal analysis. Some of them work in the frequency domain, and the other in the time domain. After their first introduction, a number of refinements was proposed for each of them. These refinements are explained in detail in the following paragraphs.

Power Spectral Density Analysis (PSD)

This method is widely used for assessing the fractal properties of time series, and works on the basis of the periodogram obtained by the Fast Fourier Transform algorithm. The relation of Mandelbrot and van Ness (1968) can be expressed as follows in the frequency domain:

$$S(f) \propto 1/f^{\beta} \tag{2}$$

where f is the frequency and $S(f)$ the correspondent squared amplitude. β is estimated by calculating the negative slope ($-\beta$) of the line relating log (S(f)) to log f (Fig. 1). Obtaining a well-defined linear fit in the double logarithmic plot is an important indication of the presence of long-range correlation in the original series.

According to Eke et al. (2000), PSD allows distinguishing between fGn and fBm series, as fGn corresponds to β exponents ranging from –1 to +1, and fBm to exponents from +1 to +3. β can be converted into $\hat{H}$ according to the following equations:

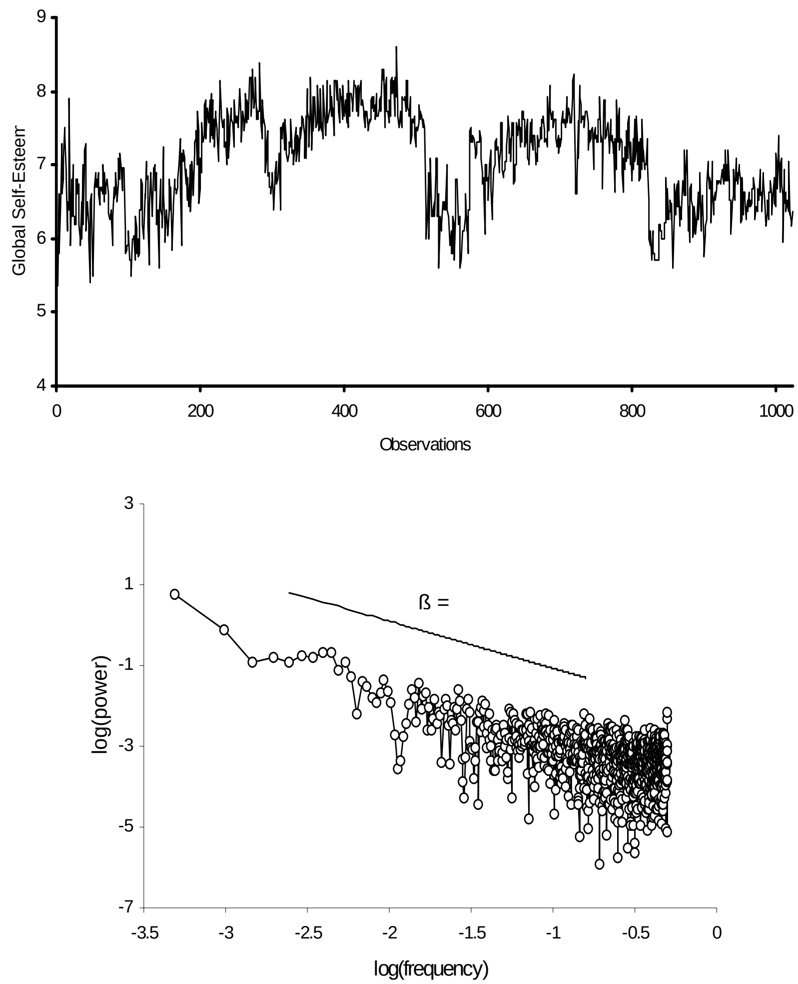

Fig. 1: Up : an example of empirical time series, obtained by a bi-daily assessment of self-esteem during 512 consecutive days (data from Delignières, Fortes & Ninot, 2004). Bottom: bi-logarithmic power spectrum obtained by PSD from the above series.

$$\hat{H} = \frac{\beta + 1}{2} \quad \text{for fGn,} \tag{3a}$$

or

$$\hat{H} = \frac{\beta - 1}{2} \quad \text{for fBm.} \tag{3b}$$

Note that in these equations and thereafter in the text, $\hat{H}$ represents the estimate provided by the analysis, and H the true exponent of the series.

Eke et al. (2000) proposed an improved version of PSD, using a combination of preprocessing operations: first the mean of the series is subtracted from each value, and then a parabolic window is applied: each value in the series is multiplied by the following function:

$$W(j) = 1 - \left(\frac{2j}{N+1} - 1\right)^2 \quad \text{for j = 1, 2, …, } N. \tag{4}$$

Thirdly a bridge detrending is performed by subtracting from the data the line connecting the first and last point of the series. Finally the fitting of β excludes the high-frequency power estimates (f > 1/8 of maximal frequency). This method was proven to provide more reliable estimates of the spectral index β, and was designated as $^{low}PSD_{we}$.

Detrended Fluctuation Analysis (DFA)

This method was initially proposed by Peng et al. (1993). The $x(t)$ series is integrated, by computing for each t the accumulated departure from the mean of the whole series:

$$X(k) = \sum_{i=1}^{k} \left[x(i) - \bar{x} \right] \tag{5}$$

This integrated series is divided into non-overlapping intervals of length n. In each interval, a least squares line is fit to the data (representing the trend in the interval). The series $X(t)$ is then locally detrended by subtracting the theoretical values $X_n(t)$ given by the regression. For a given interval length n, the characteristic size of fluctuation for this integrated and detrended series is calculated by:

$$F = \sqrt{\frac{1}{N} \sum_{k=1}^{N} \left[X(k) - X_n(k) \right]^2} \tag{6}$$

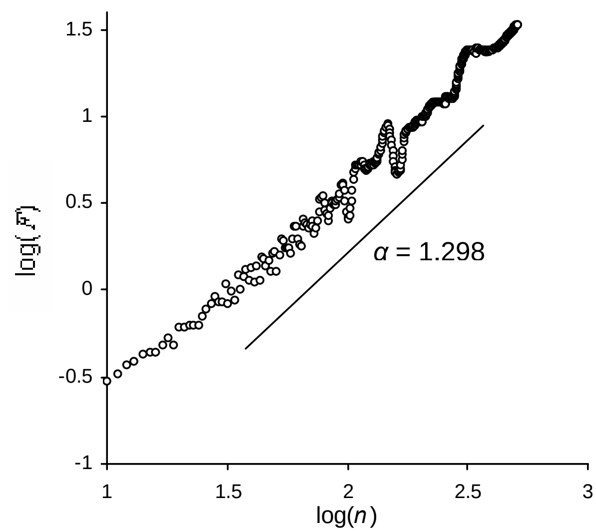

Fig. 2: Diffusion plot obtained by the application of DFA to the series presented in the upper panel of Figure 1. The statistics *F*(*n*) is plotted against the length of time intervals (see text for details).

This computation is repeated over all possible interval lengths (in practice, the shortest length is around 10, and the largest *N*/2, giving two adjacent intervals). Typically, *F* increases with interval length *n*. A power law is expected, as

$$F \propto n^{\alpha}, \tag{7}$$

where α is expressed as the slope of a double logarithmic plot of *F* as a function of *n* (Fig. 2). As PSD, DFA allows distinguishing between fGn and fBm series, fGn corresponding to α exponents ranging from 0 to 1, and fBm to exponents from 1 to 2. α can be converted into $\hat{H}$ according to the following equations:

$$\hat{H} = \alpha \quad \text{for fGn,} \tag{8a}$$

or

$$\hat{H} = \alpha - 1 \qquad \text{for fBm.} \tag{8b}$$

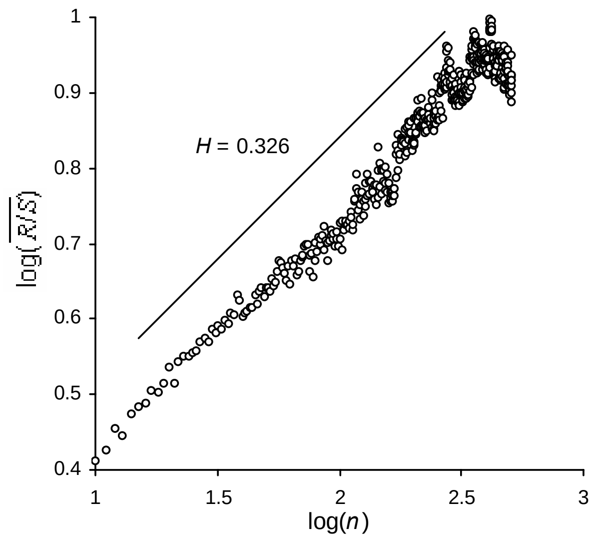

Fig. 3. Diffusion plot obtained by the application of R/S analysis to the series presented in the upper panel of Fig. 1. The statistics *R/S* is plotted against the length of time intervals (see text for details).

Rescaled Range Analysis (R/S)

This method was originally developed by Hurst (1965). The $x(t)$ series is divided into non-overlapping intervals of length n. Within each interval, an integrated series $X(t, n)$ is computed:

$$X(t,n) = \sum_{k=1}^{t} [x(k) - \bar{x}], \tag{9}$$

where $\bar{x}$ is the average within each interval. In the classical version of R/S analysis, the range R is computed for each interval, as the difference between the maximum and the minimum of integrated data $X(t, n)$.

$$R = \max_{1 \le t \le n} X(t, n) - \min_{1 \le t \le n} X(t, n) \tag{10}$$

An improved version, *R/S-detrended,* was proposed by Caccia et al. (1997): a straight line connecting the end points of each interval is subtracted from each point of the cumulative sums $X(t, n)$ before the calculation of the local range. In both methods, the range is then divided for normalization by the local standard deviation (*S*) of the original series *x(t)*. This computation is repeated over all possible interval lengths (in practice, the shortest length is around 10, and the largest (*N*-1)/2, giving two adjacent intervals). Finally the rescaled ranges *R/S* are averaged for each interval length *n*. $\overline{R/S}$ is related to *n* by a power law:

$$\overline{R/S} \propto n^{H} \qquad (11)$$

$\hat{H}$ is expressed as the slope of the double logarithmic plot of $\overline{R/S}$ as a function of *n* (Fig. 3). R/S analysis is theoretically conceived to work on fGn signals, and should provide irrelevant results for fBm signals.

Dispersional Analysis (Disp)

This method was introduced by Bassingthwaighte (1988). The *x*(*t*) series is divided into non-overlapping intervals of length *n*. The mean of each interval is computed, and then the standard deviation (*SD*) of these local means, for a given length *n*. These computations are repeated over all possible interval lengths. *SD* is related to *n* by a power law:

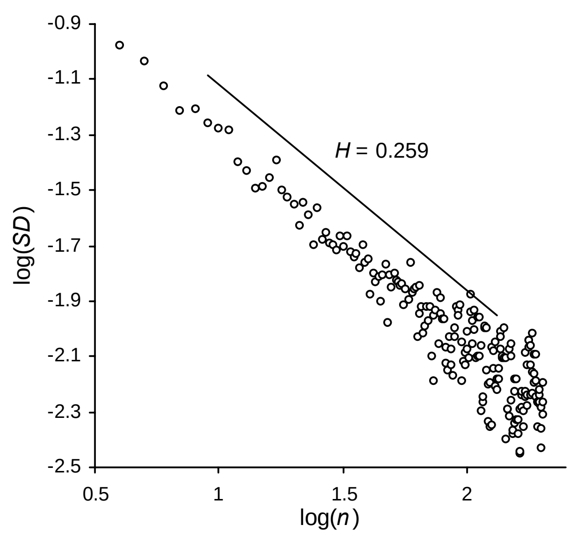

Fig. 4. Diffusion plot obtained by the application of dispersional analysis to the series presented in the upper panel of Fig. 1. The standard deviation of the mean is plotted against the length of time intervals (see text for details).

$$SD \propto n^{H\text{-}1} \tag{12}$$

The quantity (H-1) is expressed as the slope of the double logarithmic plot of *SD* as a function of *n* (Fig. 4). Obviously, the *SD'*s calculated from the highest values of *n* tend to fall below the regression line and bias the estimate. Caccia et al. (1997) suggested to ignore measures obtained from the longest intervals. As R/S analysis, Disp is theoretically conceived to work on fGn signals, and should provide irrelevant results for fBm signals.

Scaled Windowed Variance Method (SWV)

These methods were developed by Cannon et al. (1997). The *x(t)* series is divided into non-overlapping intervals of length *n*. Then the standard deviation is calculated within each interval using the formula:

$$SD = \sqrt{\frac{\sum_{t=1}^{n}[x(t)-\bar{x}]^2}{n-1}} \tag{13}$$

where $\bar{x}$ is the average within each interval. Finally the average standard deviation ($\overline{SD}$) of all intervals of length *n* is computed. This computation is repeated over all possible interval lengths. For a fractal series $\overline{SD}$ is related to *n* by a power law:

$$\overline{SD} \propto n^{H}, \tag{14}$$

where $\hat{H}$ is expressed as the slope of the double logarithmic plot of $\overline{SD}$ as a function of *n* (Fig. 5). Cannon et al. (1997) showed that a detrending of the series within each interval before the calculation of the standard deviation provided better estimates of *H*, especially with short series. Exploiting the diffusion properties of signals, SWV methods are conceived to work properly on fBm, but should provide irrelevant results on fGn.

SWV methods can also be used to distinguish between fGn and fBm near the $1/f$ boundary. Eke et al. (2000) proposed a method called *Signal summation conversion* method (SSC), based on the application of SWV to the cumulative sum of the original signal. If the obtained $\hat{H}$ is lower than 1 the original series is a fGn (in this case the cumulant series is the corresponding fBm). If $\hat{H}$ is higher than 1 the original series is a fBm.

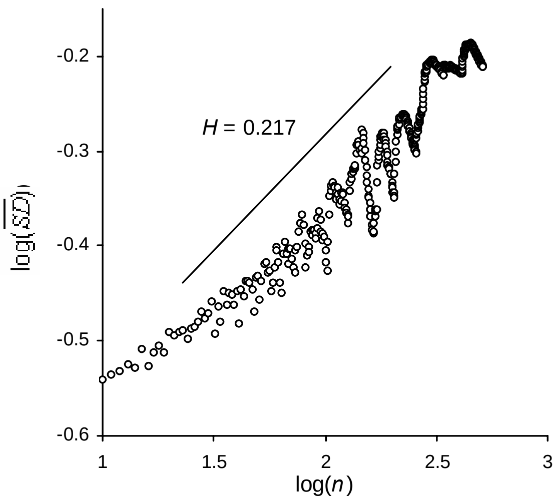

Fig. 5. Diffusion plot obtained by the application of SWV analysis to the series presented in the upper panel of Figure 1. The average standard deviation is plotted against the length of time intervals (see text for details)

IDENTIFYING FRACTAL PROCESSES

The first step, in a fractal analysis, is to detect the presence of long-range dependence in the series. All these methods theoretically allow evidencing such long-range dependences through the visual inspection of power spectrum in the frequency domain, or of the diffusion plot in the time domain. Usually, researchers apply a unique method (in the frequency domain or in the time domain), and base their conclusions on this visual, and qualitative, observation of a linear regression in double-logarithmic plots. This apparent simplicity is highly questionable and raises a number of methodological and theoretical problems. A simulated or experimental time series, while no possessing any long range correlation property, can mimic in the resultant bi-logarithmic plots the expected linear fit, and lead to false claims about the presence of underlying fractal processes (Thornton & Gilden, 2004).

Rangarajan and Ding (2000) highlighted the possible misinterpretations that could arise from the application of a unique method in fractal analysis. They developed a series of examples showing how spectral or time-related methods, applied in isolation, could lead to false identification of long-range dependence. They showed, for example, that a series composed by the superposition of an exponential trend over a white noise gives a perfect linear fit in the diffusion plot obtained through R/S analysis. The spectral method, conversely, provided a flat spectrum revealing the absence of serial correlation in the series. As well, an order-one auto-regressive process could be interpreted as a fractal series on the basis of the application of R/S analysis: the diffusion plot presents in this case also a perfect linear fit. The absence of long-range correlation is nevertheless attested by the power spectrum, with a typical flattening at low frequencies. Rangarajan and Ding (2000) concluded with the necessity of an integrated approach, based on the consistent use of several tools, in the frequency as well as in the time domain. The identification of long-range correlation requires the obtaining of the typical graphical signature with several methods, and also the consistency of the obtained slopes (this consistency is assessable through Eqs. 3 and 8).

This integrated approach, nevertheless, remains limited to the qualitative analysis of spectral and diffusion plots, and doesn’t include any test aiming at statistically evidence the presence of long-range correlation. Some authors have proposed the application of surrogate tests, in order to differentiate between long-range scaling and a random process with no long-range correlation (see, for example, Haussdorf et al., 1995). Surrogate data sets are obtained by randomly shuffling the original time series. Each surrogate data sets has the same mean and variance as the corresponding original series, and differs only in the sequential ordering. The scaling exponents of the surrogate data sets are then statistically compared to those of the original series. Nevertheless, the interest of these tests remains limited, because considering their null hypothesis, they allow to attest for the presence of correlations in the series, but they are unable to certify their long-range nature.

This problem was addressed by several recent papers (Farrell, Wagenmakers & Ratcliff, 2004; Thornton & Gilden, 2004; Wagenmakers, Farrell & Ratcliff, 2004). According to these authors, the main question is to statistically distinguish between short-term and long-term dependence in the series. Short-term dependence signifies that the current value in the series is only determined by a few number of preceding values. These short-term dependence are generally modeled by the ARMA models developed by Box and Jenkins (1976), which are composed by a combination of auto-regressive and moving-average terms. A quite simple solution could be to compare the shape of the auto-correlation function, which is supposed to be exponential in the case of a short-term memory process, and to decay according to a power law in the case of long-term dependence. This comparison, nevertheless, remains qualitative, and auto-correlation functions do not present sufficient information to give support to unequivocal statistical tests.

Wagenmakers et al. (2004) based their approach on the so-called ARFIMA models, which are frequently used in the domain of econometry for modeling long-range dependence (see, for example Diebolt and Guiraud, in press). ARFIMA is the acronym of *autoregressive fractionally integrated moving average*, and these models differ from the traditional ARMA models by the inclusion of an additional term, d, corresponding to a fractional integration process. Wagenmakers et al. (2004) proposed to test the null hypothesis $d = 0$, in order to determine whether the analyzed series belongs to the ARFIMA or to the ARMA families.

Thornton and Gilden (2004) proposed to contrast, more directly, on the basis of the obtained power spectra, short-range (ARMA) processes and long-range fractal processes. They constructed an optimal Bayesian classifier that discriminates between the two families of processes, and showed that this classifier had sufficient sensitivity to avoid false identifications.

As can be seen, the identification of true long-range correlation in a series is not so straightforward, and remains a current theoretical and methodological debate. The simple presence of a linear trend in the log-log power spectrum, or in a part of this spectrum, cannot be per se considered as a definitive proof of underlying long-range dependence.

SERIES CLASSIFICATION

The preliminary classification of series as fGn or fBm is a crucial step in fractal analysis. This procedure requires methods that can be applied to both classes of signals. Four methods, among those previously presented, satisfy this initial requirement (PSD, ${}^{low}PSD_{we}$, DFA, and SSC).

Delignières et al. (2004) analyzed the performances of these four methods, for distinguishing between fGn and fBm series apart from the $1/f$ boundary. They used simulated series with known true H exponents, generated according to the procedure proposed by Davies and Harte

(1987). Their analyses were conducted with series of fGn with "true" *H* exponents ranging from 0.1 to 0.9, by steps of 0.1, and with the corresponding series of fBm, obtained by integration. 40 fGn series and 40 fBm series were generated, for each *H* value. The accuracy of each method was assessed through the mean of the 40 estimates of *H* obtained for each true *H* value, and the variability was estimate through the standard deviation of the samples of estimates.

Their results showed that the four methods were able to distinguish between fGn and fBm, at least when true *H* exponents were sufficiently far from the 1/*f* boundary. Nevertheless, a zone of uncertainty remained: a number of series classified as fGn with exponents close to 1 were in fact fBm processes. The opposite was also observed, but to a lesser extend. This asymmetry results from the important negative bias that characterizes all methods for fBm series with low *H* exponents (i.e. $H = 0.1$ or $H = 0.2$). This negative bias was particularly salient for PSD: all fBm series with $H = 0.1$ were classified as fGn using this method. PSD worked better for $H = 0.2$, despite the negative bias, because of a low variability in *H* estimation. $^{low}PSD_{we}$ also presented a negative bias for fBm series with $H = 0.1$ or $H = 0.2$, but this bias was lesser than for PSD. DFA gave quite similar results in terms of misclassification percentages, because of a global negative bias for fBm series and a rather high variability in *H* estimation. Finally SSC appeared unable to provide a better signal classification in this uncertainty range.

This difficulty to distinguish between fGn and fBm around the 1/*f* boundary is problematic, as a number of empirical series produced by psychological or behavioral systems falls into this particular range (e.g. Delignières, Fortes, et al., 2004; Delignières, Lemoine, et al., 2004; Gilden et al., 1995; Gilden, 2001; Hausdorff et al., 1997). Finally, the best solution when series fall into this uncertainty range could be to restrain analyses to methods insensitive to the fGn/fBm dichotomy, such as $^{low}PSD_{we}$ or DFA. In other terms, the solution could be to work directly on β or α exponents, without trying to convert them into *H* metrics. This could be necessary, for example, when the goal is to determine the mean fractal exponent of a sample of series, and when some series are classified as fGn, and the others as fBm (see, for example, Delignière et al., 2004). The mean exponent can be computed in this case on the basis of the samples of β or α obtained by $^{low}PSD_{we}$ or DFA, and then possibly converted into *H*. DFA seems preferable in this case, as this method presents lower biases than spectral analyses. The high variability of DFA should be compensated by a sufficient number of series in the sample.

ESTIMATING *H*

After the classification of the series as fGn or fBm, the next step is to accurately estimate the fractal exponent. Delignières, Ramdani et al. (2004) showed that the accuracy of each method depended on the class of analyzed series (i.e. fGn or fBm), and within each class, on the localization of the underlying exponent with the H continuum (i.e. anti-persistent or persistent series).

Estimating *H* for fGn Series

When a series is clearly classified as fGn, a number of methods are available for a more accurate estimation of its fractal exponent. Clearly the least biased method for fGn series is DFA. Alternatively, one could use SWV methods on the cumulative sum of the original series. For these two methods, the bias remains limited over the whole range of H, and variability seems acceptable, at least for $H \leq 0.5$.

R/S analysis presents a positive bias for series with $H < 0.4$, but limited biases for $H \geq 0.5$, and a low variability within this range. Disp could also be proposed for the analysis of fGn series with $H \leq 0.5$. Nevertheless the level of variability seems higher for Disp than for DFA or SWV within this H range. The results of Delignières, Ramdani et al. (2004) concerning Disp are different from those of Eke et al. (2000), who selected Disp as the most relevant for the analysis of fGn series. Caccia et al. (1997) proposed improved versions of this method, reducing bias and variance in H estimation. Future tests could show if they could constitute an alternative to SWV and R/S methods for fGn series.

In conclusion, the accurate estimation of H should follow different ways according to the nature of the series. For antipersistent noises ($H < 0.5$), the best strategy seems to calculate the cumulative sum of the series, and then to apply SWV. For persistent noises, R/S analysis provides the best results.

Estimating H for fBm Series

Clearly the best method for fBm series is SWV: biases are limited over the whole range of H values, and variability remains low, especially for $H < 0.5$. In contrast, DFA presents a systematic negative bias and a high level of variability. ${}^{low}PSD_{we}$ could represent an interesting alternative, but is characterized by higher levels of variability than SWV, and some systematic biases, for very low and very high *H* values.

MEANS COMPARISONS

In some occasion, the accurate estimation of scaling exponents is not of prior interest, and the main goal of researchers is to contrast the mean exponents obtained in two or more experimental groups (see, for example, Chen, Ding, & Kelso, 2001). The main requirement for means comparison is to obtain a low variability in *H* estimation. Limited biases can be accepted, if they do not interfere with the capability of the method to distinguish between exponents. Delignières, Ramdani et al. (2004) proposed some guidelines for such comparisons.

Means Comparisons for fGn Series

PSD seems the best candidate for fGn series. Despite a negative bias for low values of *H*, and a positive bias for high values, the variability in *H* estimation remains low, even for short series. SWV, applied on the cumulative sums of the original series, could constitute a valuable alternative when $H < 0.5$. For persistent fGn ($H > 0.5$), R/S analysis could also be used.

Means Comparisons for fBm Series

For sub-diffusive fBm series ($H < 0.5$), SWV presents the best guaranties: variability remains limited (below 0.1) and biases are absent. The choice is more difficult concerning over-diffusive fBm series ($H > 0.5$), because all methods present high levels of variability within this range. The best choice seems to be $^{low}PSD_{we}$, but the use of time series longer than 1024 point is highly recommended.

RELEVANT SERIES FOR FRACTAL ANALYSIS

A given system can be observed from diverse points of view, and then produce different kinds of series. Are some series more favorable than others for expressing the fractal properties of a system? Chen, Ding and Kelso (1997) analyzed the series obtained in a synchronization tapping paradigm, during which participant had to tap in synchrony with an auditory metronome. Two series were collected: the series of the successive inter-tap intervals, and the series of successive time delays between of occurrence of the signal and the corresponding tap. The analysis of the second series revealed a clear long-range process belonging to the fGn class, with a spectral exponent β, obtained by PSD, of about 0.54, and an estimate of *H,* obtained through R/S analysis, of about 0.79. The analysis of the inter-tap intervals series gave more inconsistent results, with a positive slope in the log-log power spectrum (corresponding to a β exponent of about –1.46), and a parabolically shaped curve in the diffusion plot of R/S analysis. The authors proposed the notion of *fundamental time series*, to qualify the dynamical variables that seem to possess relevant information for revealing the fractal properties of a given system.

Another interesting example is provided by a line of research that aimed at evidencing the fractal properties of bimanual coordination. In the bimanual coordination paradigm, participants are requested to perform simultaneous rhythmical oscillations with the two hands, according to a prescribed phase relationship between the two effectors (Kelso, Holt, Rubin, & Kugler, 1981; Kelso, 1984). Two modes of coordination were proven to be particularly stable and were extensively studied: the in-phase coordination, in which homologous muscles perform simultaneous contractions, and the anti-phase coordination, in which homologous muscles perform alternate contractions. The relevant variable for analyzing such coordination is the relative phase, i.e. the difference between the instantaneous phases of each oscillator, which equals theoretically 0 degree for the in-phase mode, and 180 degrees for the anti-phase mode.

Two measures of relative phase were used in the literature, and were generally considered as interchangeable. Continuous relative phase (CRP) is derived from the position (x_t) and velocity ($\dot{x}_t$) time series of each oscillator. The phase angle is determined for each oscillator using the following equation:

$$\phi_t = \tan^{-1}(\frac{\dot{x}_t}{x_t}), \tag{15}$$

and the relative phase is determined as the instantaneous difference between the phase of each oscillator.

Discrete relative phase (DRP) is punctually computed, as the temporal difference between similar inflexion points in the oscillation of the two oscillators, reported to the period of one oscillator. CRP was often interpreted as a higher resolution form of DRP. Nevertheless, Peters, Haddad, Heiderscheit, van Emmerik and Hamill (2003) showed that these two measures essentially differ in nature: DRP yields information regarding the relative dispersion of events in oscillatory signals, while CRP described their relationship in a higher order phase space.

The fractal properties of relative phase in bimanual coordination were firstly studied by Schmidt et al. (1991) on the basis of CRP series collected in trials performed in anti-phase. Spectral analyses revealed β exponents, ranging from 1.64 to 2.96, with an average value of about 2.52. Note that this result seemed particularly unrealistic, as such β value suggested a kind of over-diffusive fBm, far from the 1/*f* range typically observed in biological systems. In a more recent study, Torre et al. (2004) analysed the fractal properties of DRP series, collected in in-phase and antiphase trials. On the basis of $^{low}PSD_{we}$ and DFA results, DRP series were classified as persistent fGns (for comparison, the mean β estimate was 0.34 for in-phase trials, and 0.44 for anti-phase trials). The estimation of *H*, performed with four methods ($^{low}PSD_{we}$, DFA, R/S analysis, and SWV), gave mean values ranging from 0.67 to 0.72 for in-phase series, and from 0.72 to 0.78 for anti-phase trials. These estimates were obviously more realistic, falling in the 1/*f* range and compatible with the essential stationarity of such behavioral series. On the basis of these results, DRP could be considered as providing a fundamental time series for assessing the fractal properties in bimanual coordination.

CRP and DRP series present another important difference. CRP series are computed as genuine time series, as successive values are spaced by equal time intervals. Conversely, DRP series correspond to a cycle-to-cycle measurement, and the time interval between two successive values depends on the local period used as denominator in the calculation of the relative phase. DRP series have to be considered as *event series*, composed of temporally ordered measures, but not as genuine time series.

In fact, most series in psychological and behavioral studies are event series, and not time series. This was the case, for example, in the experiments proposed by Gilden (2001), in which the analysed series were composed of ordered successive performances. In the special case of continuation tapping experiments, the collected series are composed of the successive inter-tap time intervals (Delignières, Lemoine, et al., 2004; Gilden et al., 1995). True time series are rather uncommon in such research (see, nevertheless, Collins & De Luca, 1993; Delignières, Fortes, et al., 2004; Treffner & Kelso, 1999).

This could be conceived as a formal obstacle for the application of time series analyses such as those previously presented. Most authors consider, nevertheless, that these time series analyses remain applicable, but obviously time cannot be considered here in its absolute sense. When applying spectral analyses, ‘frequency’ should not be read in Hertz units, but rather in inverse trial number (Gilden, 2001), or in number of cycles for N trials or observations (Musha, Katsurai & Teramachi, 1985; Yamada, 1996; Yamada & Yonera, 2001). As well, the ‘intervals’ taken into account in all time-related methods are not time intervals, but rather lengths of samples of successive observations.

This distinction between time series and event series remains crucial in fractal analyses. Researchers aiming at undertaking a fractal approach to a given system could be naturally inclined to opt for time series, considering the nature of the statistical procedures commonly used in this domain. We believe, nevertheless, that the key variable in fractal analysis is not fluctuation in time, but rather cycle-to-cycle or trial-to-trial fluctuation. In most psychological experiments, the possible mechanisms underlying dependence in series are obviously related to the serial occurrence of trials. Supposed sequential effects such as priming, knowledge of results, suggest clearly to focus on trial-to-trial variability. In other kinds of experiments (as for example, the previously evoked studies on motor coordination), the choice of discrete event series is not so directly defensible. Continuous fluctuation in time could be considered *per se* as a variable of interest, and often the dynamics of the system under study doesn't present the necessary periodical key events justifying the collection of discrete series (see, for example, Collins & De Luca, 1993, Treffner & Kelso, 1999). Frequently, nevertheless, the behavior of the system possesses a kind of periodicity that could allow collecting a discrete event series, with one measure for each successive cycle.

As explained at the beginning of this paper, the main aim of fractal analyses is to offer a renewed approach of the classical problems of stability, variability, and flexibility in the behavior of complex systems. When such a system exhibits a cyclical activity, each cycle can be considered as a functional unit, and then cycle-to-cycle fluctuation represents a kind of 'functional variability', directly related to the goal of the manifested activity. In contrast, continuous fluctuation corresponds to an 'absolute variability', which could be meaningless at the macroscopic scale of successive cycles.

THE SPECIAL CASE OF BOUNDED SERIES

Another problem that can be highlighted raises from the bounded character of the dynamics of most behavioral variables. It's important to keep in mind that a pure fBm is typically unbounded: The fluctuations grow with the time interval length in a power-law way, and the expected displacement increases indefinitely with time. In other words, the diffusion with time of a pure fBm is unlimited. In contrast, behavioral time series are generally bounded within physiological limits. This is the case, for example, for the trajectory of the center of pressure during postural sway, which is obviously bounded within the area of support of the subject's feet. As a consequence, the diffusion process remains limited and the variance of such behavioral time series cannot exceed a ceiling value, and, at least beyond a critical time interval (necessary to reach this ceiling value), should become more or less independent to time. This should naturally yield to a crossover phenomenon in the relationship between variance (or displacement) and time interval, with persistence at short time intervals and anti-persistence at long time intervals. Several experiments showed evidence of such results in the fractal analysis of biological time series (Collins & De Luca, 1993; Treffner & Kelso, 1995, 1999) and one could hypothesize that this typical feature could be due to the bounded character of the series under study. This hypothesis was considered by Liebovitch and Yang (1997), who showed that a simulated bounded random walk yields similar results, with comparable crossover phenomena.

An elegant solution for this problem is to study the fractal properties of the integrated time series, rather than those of the original signals (Feder, 1988; Hurst, 1965; Peng, Havlin, Stanley, & Goldberger, 1995). If the original signal is constrained within physiological boundaries, the integrated series is not bounded and exhibits fractal properties that can be quantified on the basis of Eq. 1. Such a procedure allows distinguishing between the uncorrelated noise (which gives a Brownian motion after integration), and a bounded fBm, which should exhibit after integration a higher diffusion than Brownian motion. In others words, the solution is to infer the fractal properties of the original signal from the diffusion properties of its integrated series. Delignières, Deschamps, Legros, and Caillou (2003) showed that the application of such method on postural data avoided the appearance of a crossover phenomenon in the diffusion plot, and gave a more readable picture of the fractality of the system.

SERIES LENGTH

Eke et al. (2000; 2002) showed that the accuracy of the estimation of fractal exponents is directly related to the length of the series. One of their main conclusions is that fractal methods cannot give reliable results with series shorter than 2^{12} data points, and in their papers, especially devoted to physiological research, they focused on results obtained with very long series (2^{17} data points). Such series cannot be collected in psychological research. The application of time series analyses supposes that the system under study remains unchanged during the whole window of observation, and in psychological experiments, the lengthening of the task raises evident problems of fatigue or lack of concentration (Madison, 2001). Generally, the use of series of 2^9 or 2^{10} data points was considered as an acceptable compromise between the requirements of time series analyses and the limitations of psychological experiments (see, for example, Chen et al., 1997, 2001; Delignières, Lemoine, et al., 2004; Gilden, 1997, 2001; Musha et al., 1985; Yamada, 1996; Yamada & Yonera, 2001; Yamada, 1995).

In their evaluation of fractal methods, Delignières, Ramdani et al. (2004) analyzed the performance of each method with particularly short series (i.e., 1024, 512, and 256 data points). They expected to find a dramatic increase of biases and variability with series shorter than 1024 data points. These results were generally present, but with rather moderate amplitudes. Only ${}^{low}PSD_{we}$ appeared severely affected by the shortening of series. This observation is very important, because of the difficulty to obtain long time series in psychological and behavioral experiments. These results suggests that a better estimate of *H* could be obtained, with a similar time on the experimental task, from the average of four exponents derived from distinct 256 data points series (with an appropriate period of rest between two successive sessions), than from a single session providing 1024 data points. This conclusion could open new perspectives of research in areas that were until now reticent for using this kind of analyses.

RESEARCH GOALS

As a conclusion, we would like to express some guidelines about the scientific goals that could be pursued through fractal analyses. Often the aim of researchers is limited to evidencing the fractal character of fluctuation in the behavior of the system under study, and the papers are concluded by a general discussion about the potential interest of dynamical systems theory and/or self-organized criticality for a renewal of the models in the concerned domain.

A first problem that has to be explored is the reliability of the typical exponents provided by fractal methods. This is not only a psychometrical question (how reliable is the measure?), but a fundamental theoretical debate: can we consider the fractal properties of a time series an inherent characteristics of the system that produced the series, or are the obtained exponents only the reflection of a temporary emergent organization, which could be entirely different from one realization of the task to the other? This question was never clearly addressed in the literature, and generally a unique assessment of fractal exponent is performed for each participant. One of us recently tried to assess the reproducibility of the fractal exponents obtained in continuous tapping experiments (Lemoine, 2004). Participant had to perform continuous tapping according to four initially prescribed tempi, and the experiment was replicated after a delay of 3 months. Results showed a good reproducibility of fractal measures, especially for the highest tempi. On the other hand, fractal exponents appeared rather specific to each experimental condition (initial tempo), suggesting a close link between task requirements and fractal properties.

A second important research goal is to identify the experimental factors susceptible to alter the fractal properties of the collected series. We evoked in the introduction that a number of studies showed significant differences in fractal exponents when contrasting young and healthy participants with elderly or diseased patients (Gottschalk et al., 1995; Hausdorff et al., 1997; Peng et al., 1995; Yoshinaga et al., 2000). Generally aging or disease led to specific alterations of $1/f$ noise, in the direction of white noise, or conversely in the direction of Brownian motion. Such alteration was interpreted as a disruption of the optimal compromise between order and chaos established by $1/f$ fractality, leading to a dramatic decrease of the adaptive capabilities of the system.

A more instructive line of research should aim at analyzing the effects of experimental factors on fractality in repeated measure designs. The goal of such a strategy is to highlight the possible temporary, or more definite alterations of fractality that could arise from the controlled manipulation of experimental factors. Such an experiment was recently conducted by Chen et al. (2001; see also Ding, Chen & Kelso, 2002), by contrasting the exponents obtained by the same participants in two tapping conditions: a condition in which the taps were performed in synchrony with the beeps of the metronome, and a condition in which the tap were in syncopation with the beeps. Their results evidenced a significant difference between the two conditions, with a spectral exponent β of about 0.54 for synchronization, and 0.77 for syncopation. It is important to note that syncopation was previously proven to be intrinsically less stable than synchronization, and more difficult to perform. Moreover, the authors showed that it was possible to shift the value of the syncopation exponents towards the values observed for synchronization by inducing a conscious strategy of coordination. One of these strategies was to produce an oscillation of the finger twice faster than the metronome, with one ‘mimicked’ tap on the beep, and one real tap off the beep. In this condition, the authors obtained exponent around 0.48, close to those observed in synchronization. In the same vein Torre et al. (2004) showed that the series of relative phase collected in bimanual coordination were characterized by significantly different exponents, according to the required coordination: The estimation of *H*, performed with four methods ($^{low}PSD_{we}$, DFA, R/S analysis, and SWV), gave mean values ranging from 0.67 to 0.72 for in-phase series, and from 0.72 to 0.78 for anti-phase trials.

These two experiments suggest that fractal exponents could be sensitive to the characteristics of the task to perform (and especially its difficulty), and may also be altered by cognitive manipulations. This kind of result opens an interesting window, for a reappraisal of a number of classical concepts, including effort, activation, concentration, fatigue, and obviously learning.

CONCLUSION

Monofractal analyses were for a long time considered as a family of rather simple statistical tools, leading to intriguing results. Recent theoretical and methodological progresses showed that the application of these methods was not so easy, and necessitated methodological and statistical attention for producing effective results. The way seems now open to a more controlled use of such methods in psychological and behavioral research, in order to undertake a real experimental research program exploiting the fractal properties of most psychological and behavioral variables.

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